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Exact equations for analysing thickness-twist trapped-energy modes in monolithic filters

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SUMMARY

Thickness-twist vibrations with energy trapping in a monolithic filter consisting of an infinite piezoceramic plate with N infinitesimally thin electrodes evaporated on to each face are analysed. By applying the Fourier transform technique, the linear three-dimensional equations for a piezoceramic plate are reduced to integral equations for the charge distributions on the electrodes. An approach for solving these equations and numerical results for a dual are given.

1. Introduction

In recent years attempts have been made to compute the eigenfrequencies and admittances of monolithic filters. A monolithic filter consists of a piezoelectric plate with two or more electrodes evaporated on to both faces in such a way that a symmetrical arrangement with respect to the middle plane results. AT-cut quartz plates or piezoceramic plates polarized in their plane or perpendicular to the faces are usually employed. A thickness vibration is set up in the plate by electrical excitation.

In a certain range of values of the exciting frequency the energy stored in the plate is mainly concentrated in the volumes between corresponding electrodes on the faces. This phenomenon is called energy trapping and is due to both the mechanical mass loading caused by the electrodes and the piezoelectric effect ([1]–[3]). Monolithic filters are excited in this range of frequencies.

No rigorous analysis of the vibrations of these filters has appeared to date. Only approximating treatments of quartz and piezoceramic filters which are infinitely extended in at least one direction and partly covered by infinite strip electrodes are known. In quartz filters the mass loading is the most important effect causing energy trapping. The treatments of this kind of filters have been based on the elastic equations; the piezoelectricity has been neglected ([4], [5]). In these investigations thickness-shear and thickness-twist modes were considered. Using the Mindlin-approximations, ordinary differential equations have been derived for the electroded parts and the parts without electrodes. In addition, boundary conditions at the edge of the plate have been stated as well as continuity conditions at the interfaces of the electroded and unelectroded regions. The eigenfrequencies determined by the above-mentioned equations and conditions are only accurate if the widths of the electrodes and "gaps" between the electrodes are large with respect to the thickness of the plate.

In piezoceramic plates the piezoelectricity must be taken into account and the mass loading is less important. Approximate resonant frequencies and capacitances have also been computed for piezoceramic plates in which a thickness-twist or a thickness-dilation vibration with energy trapping is excited ([3], [6]). In [3] a single resonator has been considered, consisting of a piezoceramic plate of infinite extent in all directions with one infinite strip electrode evaporated on to each face. Wave solutions of the above-mentioned modes have been given for the electroded and unelectroded parts. By coupling these solutions at the interfaces of these parts, resonance spectra and capacitances have been computed. In [6] the analysis for thickness-twist has been extended to a filter with N pairs of electrodes. Since the mechanical continuity

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conditions are only roughly satisfied at the interfaces and the electrical ones not at all, the results of these treatments will be only reliable for large values of the ratio's widths of the electroded and unelectroded parts over the thickness of the plate.

The present author has given an exact approach ([7]) for analysing trapped energy modes in infinite piezoceramic plates in case mass loading can be neglected, i.e. if the electrodes can be assumed to be infinitesimally thin. A plate polarized in its plane has been considered. Both faces were covered by infinite strip electrodes running parallel to the polarization. By a potential difference between the electrodes thickness-twist waves have been excited. Applying the Fourier transform technique to the piezoelectric equations for the complete plate, an integral equation for the charge distribution on the electrodes has been derived, which was then solved numerically. Correct resonant frequencies for all values of the ratio electrode-width over plate-thickness have been obtained. These computations confirmed the presumption that the results given in [3] and [6] are inaccurate for values of that ratio which are not much larger than 1.

In this paper the Fourier transform technique is applied to thickness-twist vibrations in infinite piezoceramic plates covered by an arbitrary finite number of electrodes. Again an integral equation is obtained for the charge distributions on the electrodes. The integral form is simplified substantially by using the residue theorem for complex functions. The resulting equation can be solved by dividing the electrode range into subintervals and by approximating the charge distribution by a quadratic polynomial in each subinterval.

Numerical results are given for a dual consisting of a plate with two pairs of electrodes on the faces. Resonant frequencies are computed for a fixed value of the electromechanical coupling factor and a number of values of the width of the electrodes and of the gap between the electrodes. Resonances are also given for a fixed geometry and numerous values of the coupling factor.

2. Formulation of the problem

We consider an infinitely extended piezoceramic plate of constant thickness $2h$. The plate is assumed to be uniformly polarized in its plane. Figure 1 shows a cross section of a part of the plate. We choose Cartesian coordinates (x_1, x_2, x_3) with $x_2 = \pm h$ defining the faces of the plate. The x_3-axis is in the direction of polarization. Both faces are covered by N infinitesimally thin strip electrodes. The pair of electrodes n, $n=1, \ldots, N$, occupies the regions $x_2 = \pm h$, $a_n \leqq x_1 \leqq b_n$.

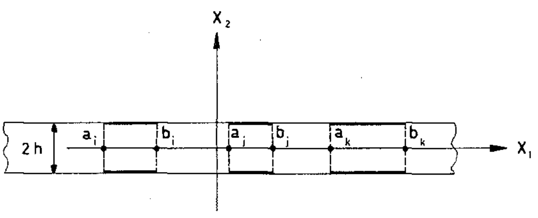

Figure 1. Cross section of a part of the filter.

The faces are free of stresses. A periodic vibration is set up in the plate by electrical excitation. The potential of the electrode of pair n on the boundary $x_2 = h$ is denoted by $V_n \cdot \exp(i\omega t)$ and the total charge by $Q_n \cdot \exp(i\omega t)$. Here ω represents the circular frequency and t the time. We assume that either the potential or the charge are prescribed on a pair of electrodes in such a way that the lower electrode has a potential $-V_n \cdot \exp(i\omega t)$ if V_n is given or a charge $-Q_n \cdot \exp(i\omega t)$ if Q_n is given. At least one of the prescribed values V_n, Q_n, $n=1, \ldots, N$, will not vanish. In the remainder the exponential time factor is omitted.

Since the plate is polarized in the x_3-direction, a standing thickness-twist mode with a particle displacement U_3 in the x_3-direction is excited. We assume that U_3 is a function of x_1 and x_2. Moreover, a potential $V(x_1, x_2)$ will exist both inside and outside the plate. The displacements U_1 and U_2 in the x_1- and x_2-direction, respectively, are supposed to vanish. The only non-zero stresses and electric displacements expressed in derivatives of U_3 and V are now ([7]),

$$T_{\alpha 3} = ({}^E c_{1313} U_3 + e_{113} V)_{,\alpha}, \tag{2.1a}$$

$$D_\alpha = (e_{113} U_3 - {}^S\varepsilon_{11} V)_{,\alpha}, \tag{2.1b}$$

where α is 1 or 2. In (2.1) the tensor notation with respect to (x_1, x_2, x_3) is employed. $T_{\alpha 3}$ are shear stresses, D_α components of the electric displacement, ${}^E c_{1313}$ is an elastic constant measured at constant electric field, e_{113} a piezoelectric constant and ${}^S\varepsilon_{11}$ a dielectric constant measured at constant strain. A comma followed by an index α denotes differentiation with respect to x_α.

The $T_{\alpha 3}$ have to satisfy the equation of motion

$$T_{13,1} + T_{23,2} = \mu\omega^2 U_3, \tag{2.2a}$$

where μ denotes the mass density and D_α the Maxwell equation

$$D_{1,1} + D_{2,2} = 0. \tag{2.2b}$$

We assume that outside the plate the electrostatic equations of vacuum hold. Hence, in addition to (2.2b), we have

$$D_\alpha = -\varepsilon_0 V_{,\alpha}, \tag{2.3}$$

where ε_0 is the permittivity of free space.

At $x_2 = \pm h$ the following transition conditions of electrostatics are valid:

$$V^{(1)}_{,1} - V^{(2)}_{,2} = 0, \tag{2.4a}$$

$$D^{(1)}_2 - D^{(2)}_2 = F. \tag{2.4b}$$

Here F represents the charge density on the electrodes.

By means of symmetry considerations a half-space problem is formulated. Let us denote the solution for the plate and its environs by

$$\{U_3, V\}(x_1, x_2). \tag{2.5}$$

We thus assume that (2.5) satisfies equations (2.1)–(2.4) and the boundary conditions.

We now introduce the coordinates $(x_1, x_2' = -x_2, x_3' = -x_3)$. With respect to these coordinates the elastic and dielectric constant are the same ones as in (x_1, x_2, x_3), while the piezoelectric constant become the opposite value. Hence the equations (2.1)–(2.4) are also valid in the coordinates (x_1, x_2', x_3'), provided e_{311} is replaced by $-e_{311}$. The solution $\{U_3, -V\}(x_1, x_2')$ satisfies these equations in the new coordinates. This solution yields for the electrodes on the boundary $x_2 = h$ corresponding with $x_2' = -h$ a potential $-V(x_1, -h)$ and a charge density $-F(x_1, -h)$. If the potential is prescribed on the electrodes n, we have

$$-V(x_1, -h) = -(-V_n) = V_n, \tag{2.6a}$$

and if the charge is given,

$$-\int_{a_n}^{b_n} F(x_1, -h)\,dx_1 = -(-Q_n) = Q_n. \tag{2.6b}$$

Consequently, also, the boundary conditions are satisfied.

Substitution of $x_2' = -x_2$, $x_3' = -x_3$ in the latter solution and simple tensor calculus yield

$$\{-U_3, -V\}(x_1, -x_2). \tag{2.7}$$

Since (2.5) and (2.7) represent the same solution, we obtain the condition

$$U_3 = V = 0, \quad x_2 = 0. \tag{2.8}$$

In virtue of (2.8) we can restrict ourselves to the half-space $x_2 \geqq 0$.

3. Application of the Fourier transform technique

The mixed boundary value problem formulated in the preceding section is considered in the trapped energy range. Hence we confine ourselves to the frequency interval

$$\omega_e < \omega < \omega_u \,. \tag{3.1}$$

Here ω_e represents the cut-off frequency of the thickness-twist wave propagation in an infinite plate which is completely covered by electrodes, and ω_u the corresponding frequency in an infinite plate without electrodes. In virtue of the restriction (3.1) the energy per unit length along the x_3-axis is mainly concentrated in the regions $a_n \leqq x_1 \leqq b_n$ of the plate. The amplitudes of the unknowns vanish exponentially as $|x_1| \to \infty$. Substitution of (2.1) into (2.2) yields the differential equations

$$^{D}c_{1313}\,\Delta U_3 + \mu\omega^2 U_3 = 0\,, \tag{3.2a}$$

$$\Delta\Phi = 0\,. \tag{3.2b}$$

Here Δ represents the Laplace operator in the coordinates x_1 and x_2, $^{D}c_{1313}$ is an elastic constant measured at constant electric displacement,

$$^{D}c_{1313} = {}^{E}c_{1313} + \frac{(e_{113})^2}{^{S}\varepsilon_{11}} \tag{3.3}$$

and the function Φ is defined by

$$\Phi = e_{113}\,U_3 - {}^{S}\varepsilon_{11}\,V\,. \tag{3.4}$$

We denote the Fourier transform of a function $f(x_1, x_2)$ by $f^*(\xi, x_2)$, hence ([8]),

$$f^*(\xi, x_2) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(x_1, x_2)\,e^{i\xi x_1}\,dx_1\,. \tag{3.5}$$

Multiplying both sides of (3.2) by $(2\pi)^{-\frac{1}{2}}e^{i\xi x_1}$ and integrating over the whole range of x_1, we arrive at the following ordinary differential equations,

$$^{D}c_{1313}(-\xi^2 U_3^* + U_{3,22}^*) + \mu\omega^2 U_3^* = 0\,, \tag{3.6a}$$

$$-\xi^2\Phi^* + \Phi^*_{,22} = 0\,. \tag{3.6b}$$

Taking into account condition (2.8) the solution of (3.6) reads

$$U_3^* = A\sinh\zeta x_2\,, \tag{3.7a}$$

$$\Phi^* = e_{113}B\sinh\xi x_2\,, \tag{3.7b}$$

where A and B are arbitrary functions of ξ and ζ is given by

$$\zeta = \left(\xi^2 - \frac{\mu\omega^2}{^{D}c_{1313}}\right)^{\frac{1}{2}}. \tag{3.8}$$

In order that ζ is a single-valued function of ξ, the complex ξ-plane is cut along the part of the real ξ-axis between the two branch points and that branch is chosen which tends to ξ as $\xi \to \pm\infty$.

Since the faces are free of stresses, (2.1a) yields

$$A\zeta\cosh\zeta h - k^2 B\xi\cosh\xi h = 0\,. \tag{3.9}$$

The quantity k represents an electromechanical coupling factor,

$$k = \frac{e_{113}}{(^{D}c_{1313}\,{}^{S}\varepsilon_{11})^{\frac{1}{2}}}\,. \tag{3.10}$$

Transforming the equations (2.1b) and (3.4) we obtain the relations

$$A\sinh\zeta h - B\sinh\xi h = \frac{^{S}\varepsilon_{11}\,V^*(\xi, h)}{e_{113}}\,, \tag{3.11a}$$

$$e_{113} B\xi \cosh \xi h = D_2^*(\xi, h). \tag{3.11b}$$

Combination of (3.9) and (3.11) yields

$$D_2^*(\xi, h) = -{}^S\varepsilon_{11} \xi H(\xi) V^*(\xi, h), \tag{3.12}$$

where

$$H(\xi) = \left(\frac{\sinh \xi h}{\cosh \xi h} - k^2 \frac{\xi \sinh \zeta h}{\zeta \cosh \zeta h}\right)^{-1}. \tag{3.13}$$

The region $x_2 > h$ is governed by

$$\Delta V = 0. \tag{3.14}$$

From (3.14) and (2.3) we derive for the outer region

$$D_2^*(\xi, h) = \varepsilon_0 |\xi| V^*(\xi, h). \tag{3.15}$$

Combining (3.12) and (3.15) we obtain the relation

$$F^*(\xi) = \{\varepsilon_0 \operatorname{sign} \xi + {}^S\varepsilon_{11} H(\xi)\} \xi V^*(\xi, h), \tag{3.16}$$

where $F^*(\xi)$ represents the Fourier transform of the charge density $F(r)$ on the electrodes,

$$F^*(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{n=1}^{N} \int_{a_n}^{b_n} F(r) e^{i\xi r} dr. \tag{3.17}$$

The inverse Fourier transform yields

$$V(x_1, h) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} V^*(\xi, h) e^{-i\xi x_1} d\xi. \tag{3.18}$$

From (3.16), (3.17) and (3.18) we derive the equation

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x_1}}{\{\varepsilon_0 \operatorname{sign} \xi + {}^S\varepsilon_{11} H(\xi)\} \xi} \sum_{n=1}^{N} \int_{a_n}^{b_n} F(r) e^{i\xi r} dr\, d\xi = 2\pi V(x_1, h). \tag{3.19}$$

The charge density $F(r)$ is determined by subjecting this integral equation to the electrical boundary conditions.

We now introduce the non-dimensional quantities

$$x = \frac{x_1}{h}, \quad \rho = \frac{r}{h}, \quad \eta = h\xi, \quad \lambda = h\zeta,$$

$$A_n = \frac{a_n}{h}, \quad B_n = \frac{b_n}{h}, \quad \Psi_n = \frac{V_n}{V_0}, \tag{3.20}$$

$$\Omega = \frac{h}{\pi} \left(\frac{\mu}{{}^D c_{1313}}\right)^{\frac{1}{2}} \omega, \quad G(\rho) = \frac{h}{2\pi {}^S\varepsilon_{11} V_0} F(h\rho),$$

where V_0 is an arbitrary potential. Using (3.20) and denoting the union of the intervals $[A_n, B_n]$, $n = 1, \ldots, N$, by L, we can write (3.19) in the form

$$\int_{-\infty}^{\infty} K(\eta) e^{-i\eta x} \int_L G(\rho) e^{i\eta\rho} d\rho\, d\eta = \Psi(x), \qquad x \in L, \tag{3.21}$$

where

$$\Psi(x) = \Psi_n, \quad A_n \leqq x \leqq B_n. \tag{3.22}$$

Neglecting the term $\varepsilon_0 / {}^S\varepsilon_{11}$, which is very small for a piezoceramic, we derive from (3.13), (3.19) and (3.20),

$$K(\eta) = \frac{\sinh \eta}{\eta \cosh \eta} - k^2 \frac{\sinh \lambda}{\lambda \cosh \lambda}. \tag{3.23}$$

For λ, defined by (3.20), we have the expression

$$\lambda = (\eta^2 - \pi^2 \Omega^2)^{\frac{1}{2}}. \tag{3.24}$$

In terms of dimensionless frequencies, the trapped energy range (3.1) becomes ([3]),

$$\Omega_e < \Omega < \tfrac{1}{2}. \tag{3.25}$$

The lower bound Ω_l, the dimensionless cutoff frequency for a fully electroded plate, equals the first positive root of the equation

$$\text{tg}\ \pi\Omega = \frac{\pi\Omega}{k^2}. \tag{3.26}$$

4. Discussion of the integral equation

The function $K(\eta)$ has simple poles, which are solutions of

$$\cosh \eta = 0, \tag{4.1a}$$

$$\cosh \lambda = 0. \tag{4.1b}$$

Equation (4.1a) yields the values

$$\eta = \pm i\pi(l+\tfrac{1}{2}), \qquad l = 0, 1, 2, \dots . \tag{4.2}$$

Solving (4.1b) with λ given by (3.24), we obtain

$$\eta^2 = \pi^2\{\Omega^2 - (l+\tfrac{1}{2})^2\}. \tag{4.3}$$

In virtue of (3.25) this relation yields only poles on the imaginary η-axis,

$$\eta = \pm i\pi\{(l+\tfrac{1}{2})^2 - \Omega^2\}^{\frac{1}{2}}. \tag{4.4}$$

Defining

$$K(0) = 1 - k^2 \frac{\text{tg}\ \pi\Omega}{\pi\Omega}, \tag{4.5a}$$

$$K(\pi\Omega) = \frac{\sinh \pi\Omega}{\pi\Omega \cosh \pi\Omega} - k^2, \tag{4.5b}$$

$K(\eta)$ is an analytic function in the complex η-plane except at the poles on the imaginary axis given by (4.2) and (4.4). We observe that $K(\eta)$ vanishes as $(1-k^2)/\eta$ for $\eta \to \pm\infty$.

Physical considerations permit the assumption that $G(\rho)$ has a continuous derivative for $A_n < \rho < B_n$ and a square-root singularity at $\rho = A_n$ and $\rho = B_n$. We can now write

$$\int_{-C}^{C} K(\eta) e^{-i\eta x} \int_{A_n}^{B_n} G(\rho) e^{i\eta\rho} d\rho\, d\eta = \int_{A_n}^{B_n} G(\rho) \int_{-C}^{C} K(\eta) e^{i\eta(\rho - x)} d\eta\, d\rho, \tag{4.6}$$

where C is a positive constant. For C tending to infinity, the left-hand side of (4.6) tends to the left-hand side of (3.21). Since $K(\eta)$ vanishes as $O(1/\eta)$ for $\eta \to \pm\infty$,

$$\lim_{C\to\infty} \int_{-C}^{C} K(\eta) e^{i\eta(\rho - x)} d\eta \tag{4.7}$$

exists for every ρ except $\rho = x$.

For x an interior point of L, we define the subrange $[x-\delta, x+\delta]$, where δ is a positive number, by L_δ. δ is chosen so small that $x-\delta$ and $x+\delta$ are also interior points of L. If $x = A_n$ or $x = B_n$, $n = 1, \dots, N$, L_δ represents the interval $[A_n, A_n+\delta]$ with $A_n+\delta < B$, respectively the interval $[B_n-\delta, B_n]$ with $B_n - \delta > A_n$. Since (4.7) exists for $\rho \neq x$, the orders of integration in (3.21) can be interchanged for $\rho \in L - L_\delta$. Then we obtain the integral equation

$$\int_{L-L_\delta} G(\rho) \int_{-\infty}^{\infty} K(\eta) e^{i\eta(\rho-x)} d\eta\, d\rho + \int_{-\infty}^{\infty} K(\eta) e^{i\eta x} \int_{L_\delta} G(\rho) e^{i\eta\rho} d\rho\, d\eta = \Psi(x), \; x \in L. \tag{4.8}$$

An estimation of

$$\int_{-\infty}^{\infty} K(\eta)\mathrm{e}^{-i\eta x}\int_{L_\delta} G(\rho)\mathrm{e}^{i\eta\rho}\,d\rho\, d\eta \tag{4.9}$$

will now be given. It is clear that integration over a finite η-range yields a contribution of order δ as $\delta\to 0$. To estimate the integration over the remaining parts, we consider first an interior point x of L. Integration by parts yields

$$\int_{x-\delta}^{x+\delta} G(\rho)\mathrm{e}^{i\eta\rho}\,d\rho = \frac{\mathrm{e}^{i\eta x}}{i\eta} f(x,\delta)\,. \tag{4.10}$$

Denoting the derivative of G by G',

$$f(x,\delta) = [\mathrm{e}^{i\eta u} G(x+u)]_{u=-\delta}^{u=\delta} - \int_{-\delta}^{\delta} \mathrm{e}^{i\eta u} G'(x+u)\,du\,. \tag{4.11}$$

Since G' is continuous in $[x-\delta, x+\delta]$, $f(x,\delta)$ is $O(\delta)$. Hence the integrand

$$\frac{K(\eta) f(x,\delta)}{i\eta} \tag{4.12}$$

is $O(\delta/\eta^2)$ as $\eta\to\pm\infty$ and $\delta\to 0$. Consequently (4.9) converges and is $O(\delta)$ for an interior point x.

In the interval $[A_n, A_n+\delta]$, $G(\rho)$ can be written in the form

$$G(\rho) = \frac{D}{u^{\frac{1}{2}}}\{1+O(u)\}\,, \tag{4.13}$$

where $u=\rho-A_n$ and D is a constant. Applying the substitution $\eta u = v$, we obtain

$$\int_{A_n}^{A_n+\delta} G(\rho)\mathrm{e}^{i\eta\rho}\,d\rho = D\,\mathrm{e}^{i\eta x}\left\{\frac{1}{\eta^{\frac{1}{2}}}\int_0^{\eta\delta}\frac{\mathrm{e}^{iv}}{v^{\frac{1}{2}}}\,dv + O\left(\frac{\delta^{\frac{1}{2}}}{\eta}\right)\right\}. \tag{4.14}$$

Substitution of the order term in (4.9) yields a convergent integral for $\eta\to\pm\infty$, which is $O(\delta^{\frac{1}{2}})$. Substituting $\eta\delta = w$, we obtain

$$D\int_E^{\infty}\frac{K(\eta)}{\eta^{\frac{1}{2}}}\int_0^{\eta\delta}\frac{\mathrm{e}^{iv}}{v^{\frac{1}{2}}}\,dv\,d\eta = D\delta^{-\frac{1}{2}}\int_{\delta E}^{\infty}\frac{K(w/\delta)}{w^{\frac{1}{2}}}\int_0^{w}\frac{\mathrm{e}^{iv}}{v^{\frac{1}{2}}}\,dv\,dw\,, \tag{4.15}$$

where E is a positive constant. Since $K(w/\delta)$ is $O(\delta/w)$ as $w\to\pm\infty$ and

$$\int_0^{w}\frac{\mathrm{e}^{iv}}{v^{\frac{1}{2}}}\,dv \tag{4.16}$$

is bounded, the right-hand side of (4.15) converges and is $O(\delta^{\frac{1}{2}})$ as $\delta\to 0$. An identical result is obtained for $x=B_n$. In virtue of these statements the integral (4.9) vanishes for $\delta\to 0$, hence (4.8) simplifies to

$$\lim_{\delta\to 0}\int_{L-L_\delta} G(\rho)\int_{-\infty}^{\infty} K(\eta)\mathrm{e}^{i\eta(\rho-x)}\,d\eta\,d\rho = \Psi(x)\,, \qquad x\in L\,. \tag{4.17}$$

The integral over the infinite η-range can be evaluated by means of the theory of residues. Therefore we consider

$$\int_{C_l} K(\eta)\,\mathrm{e}^{i\eta(\rho-x)}\,d\eta\,. \tag{4.18}$$

The contour C_l consists of the part of the real axis between the points $\eta=\pm\pi l$, where l is a positive integer, and is closed by a semi-circle in the upper half-plane for $\rho > x$ and a semi-circle in the lower half-plane for $\rho < x$. $K(\eta)$ vanishes uniformly on the semi-circles for $l\to\infty$. Hence the contributions to (4.18) due to the integration over these arcs vanishes as $l\to\infty$ ([9]). Determining the residues of $K(\eta)\exp\{i\eta(\rho-x)\}$ at the poles (4.2) and (4.4) we obtain

$$\int_{-\infty}^{\infty} K(\eta)\,e^{i\eta(\rho-x)}d\eta = 2\pi \sum_{l=0}^{\infty}\left\{\frac{e^{-\alpha_l|\rho-x|}}{\alpha_l} - k^2\,\frac{e^{-\beta_l|\rho-x|}}{\beta_l}\right\}, \tag{4.19}$$

where

$$\alpha_l = \pi(l+\tfrac{1}{2}), \tag{4.20a}$$

$$\beta_l = \pi\{(l+\tfrac{1}{2})^2-\Omega^2\}^{\frac{1}{2}}. \tag{4.20b}$$

The sums (4.19) converge uniformly for every positive δ. Hence (4.17) transforms into

$$\lim_{\delta\to 0}\sum_{l=0}^{\infty}\int_{L-L_\delta} G(\rho)\left\{\frac{e^{-\alpha_l|\rho-x|}}{\alpha_l} - k^2\,\frac{e^{-\beta_l|\rho-x|}}{\beta_l}\right\}d\rho = \frac{1}{2\pi}\,\Psi(x), \qquad x\in L. \tag{4.21}$$

5. An approach for solving (4.21)

Equation (4.21) can be solved by dividing each interval $[A_n, B_n]$ into $2M_n$ sub-intervals, separated by points $\rho_{n,m}$, such that

$$\rho_{n,2m}-\rho_{n,2m-1} = \rho_{n,2m+1}-\rho_{n,2m} = t_{n,m}, \quad m=1,\dots,M_n. \tag{5.1}$$

In the sub-intervals $[\rho_{n,2m-1}, \rho_{n,2m+1}]$ the function $G(\rho)$ is approximated by a quadratic polynomial through the points $\rho_{n,2m-1}$, $\rho_{n,2m}$ and $\rho_{n,2m+1}$, yielding

$$G(\rho)\sim g_{n,2m-1} + \frac{\rho-\rho_{n,2m-1}}{t_{n,m}}\left(2g_{n,2m}-\tfrac{3}{2}g_{n,2m-1}-\tfrac{1}{2}g_{n,2m+1}\right)$$

$$+\frac{(\rho-\rho_{n,2m-1})^2}{2(t_{n,m})^2}\left(g_{n,2m-1}-2g_{n,2m}+g_{n,2m+1}\right), \tag{5.2}$$

where

$$g_{n,m} = G(\rho_{n,m}), \qquad n=1,\dots,N,\ m=1,\dots,2M_n+1. \tag{5.3}$$

Substituting (5.2) into (4.21), integrating which respect to ρ and requiring that the integral equation is satisfied in the points $x_{n,m}$, $n=1,\dots,N$, $m=1,\dots,2M_n+1$, we obtain the following system of linear equations,

$$\sum_{p=1}^{N}\sum_{q=1}^{N}\{w_{p,q,0}(x_{n,m})\,g_{p,2q-1} + w_{p,q,1}(x_{n,m})\,g_{p,2q} +$$

$$w_{p,q,2}(x_{n,m})\,g_{p,2q+1}\} = \frac{1}{2\pi}\,\Psi(x_{n,m}). \tag{5.4}$$

The coefficients can be written in the form

$$w_{p,q,i}(x_{n,m}) = \sum_{l=0}^{\infty}\{v_{p,q,i}(x_{n,m},\,\alpha_l) - k^2\,v_{p,q,i}(x_{n,m},\,\beta_l)\}. \tag{5.5}$$

Using the notation

$$\bar{x} = x_{n,m}, \quad \bar{\rho}_1 = \rho_{p,2q-1}, \quad \bar{\rho}_2 = \rho_{p,2q+1}, \quad \bar{t} = t_{p,q}, \tag{5.6}$$

we have in the limit $\delta\to 0$ the expressions

$$v_{p,q,0}(x_{n,m},\alpha) = \begin{cases} \dfrac{e^{-\alpha(\bar{\rho}_1-\bar{x})}\{2-3\alpha\bar{t}+2\alpha^2\bar{t}^2-e^{-2\alpha\bar{t}}(2+\alpha\bar{t})\}}{2\alpha^4\bar{t}^2}, & \bar{\rho}_1\geqq\bar{x}, \\[2ex] \dfrac{2-e^{-\alpha\bar{t}}(2+2\alpha\bar{t}+\alpha^2\bar{t}^2)}{\alpha^4\bar{t}^2}, & \rho_{p,2q}=x_{n,m}, \\[2ex] \dfrac{e^{-\alpha(\bar{x}-\bar{\rho}_2)}\{2-\alpha\bar{t}-e^{-2\alpha\bar{t}}(2+3\alpha\bar{t}+2\alpha^2\bar{t}^2)\}}{2\alpha^4\bar{t}^2}, & \bar{\rho}_2\leqq\bar{x}, \end{cases} \tag{5.7a}$$

$$v_{p,q,1}(x_{n,m},\alpha)=\begin{cases}\dfrac{2e^{-\alpha(\bar{\rho}_1-\bar{x})}\{-1+\alpha\bar{t}+e^{-2\alpha\bar{t}}(1+\alpha\bar{t})\}}{\alpha^4\bar{t}^2}, & \bar{\rho}_1\geqq\bar{x},\\[2mm] \dfrac{2\{-2+\alpha^2\bar{t}^2+2e^{-\alpha\bar{t}}(1+\alpha\bar{t})\}}{\alpha^4\bar{t}^2}, & \rho_{p,2q}=x_{n,m},\\[2mm] \dfrac{2e^{-\alpha(\bar{x}-\bar{\rho}_2)}\{-1+\alpha\bar{t}+e^{-2\alpha\bar{t}}(1+\alpha\bar{t})\}}{\alpha^4\bar{t}^2}, & \bar{\rho}_2\leqq\bar{x},\end{cases} \tag{5.7b}$$

$$v_{p,q,2}(x_{n,m},\alpha)=\begin{cases}\dfrac{e^{-\alpha(\bar{\rho}_1-\bar{x})}\{2-\alpha\bar{t}-e^{-2\alpha\bar{t}}(2+3\alpha\bar{t}+2\alpha^2\bar{t}^2)\}}{2\alpha^4\bar{t}^2}, & \bar{\rho}_1\geqq\bar{x},\\[2mm] \dfrac{2-e^{-\alpha\bar{t}}(2+2\alpha\bar{t}+\alpha^2\bar{t}^2)}{\alpha^4\bar{t}^2}, & \rho_{p,2q}=x_{n,m},\\[2mm] \dfrac{e^{-\alpha(\bar{x}-\bar{\rho}_2)}\{2-3\alpha\bar{t}+2\alpha^2\bar{t}^2-e^{-2\alpha\bar{t}}(2+\alpha\bar{t})\}}{2\alpha^4\bar{t}^2}, & \bar{\rho}_2\leqq\bar{x}.\end{cases} \tag{5.7c}$$

If the potentials on the electrodes are given, we have $\Sigma_{n=1}^{N}(2M_n+1)$ equations for an identical number of unknowns $g_{n,m}$, $n=1, \ldots, N$, $m=1, \ldots, 2M_n+1$. After computing the coefficients (5.5) we can evaluate the charge distribution on the electrodes by solving (5.4). The total charges Q_n, $n=1, \ldots, N$, are obtained from the Simpson formula

$$Q_n=\sum_{m=1}^{M_n}\frac{t_{n,m}}{3}(g_{n,2m-1}+4g_{n,2m}+g_{n,2m+1})\,. \tag{5.8}$$

An equation of the form (5.8) is added to the system (5.4) if a charge Q_n is prescribed instead of a potential V_n. In that case we also have an additional unknown V_n, resulting again into a system with an identical number of equations and unknowns.

6. Numerical results for a dual

An infinitely extended plate is considered with two electrodes on each face. Such a configuration is called a dual. The electrode regions are assumed to be symmetrical with respect to the x_2-axis; they are given by $-B\leqq x\leqq -A$ and $A\leqq x\leqq B$.

For a dual equation (4.21) reads

$$\lim_{\delta\to 0}\int_{L-L_\delta}G(\rho)f(|\rho-x|)d\rho=\frac{1}{2\pi}\Psi_1\,,\quad -B\leqq x\leqq -A\,, \tag{6.1a}$$

$$\lim_{\delta\to 0}\int_{L-L_\delta}G(\rho)f(|\rho-x|)d\rho=\frac{1}{2\pi}\Psi_2\,,\quad A\leqq x\leqq B\,, \tag{6.1b}$$

where

$$f(|\rho-x|)=\sum_{l=0}^{\infty}\left\{\frac{e^{-\alpha_l|\rho-x|}}{\alpha_l}-k^2\frac{e^{-\beta_l|\rho-x|}}{\beta_l}\right\}. \tag{6.2}$$

Substituting $x=-x'$, equation (6.1a) is transformed to the region $A\leqq x'\leqq B$. In the remainder the prime is dropped. Writing further

$$G(\rho)=\bar{G}^s(-\rho)+\bar{G}^a(-\rho)\,,\quad -B\leqq\rho\leqq -A\,, \tag{6.3a}$$

$$G(\rho)=\bar{G}^s(\rho)-\bar{G}^a(\rho)\,,\quad A\leqq\rho\leqq B\,, \tag{6.3b}$$

equations (6.1) transform into

$$\lim_{\delta\to 0}\int_{L'-L_\delta}\{\bar{G}^s(\rho)f_1(\rho,x)+\bar{G}^a(\rho)f_2(\rho,x)\}d\rho=\frac{1}{2\pi}\Psi_1\,, \tag{6.4a}$$

$$\lim_{\delta\to 0}\int_{L'-L_\delta}\{\bar{G}^s(\rho)f_1(\rho,x)-\bar{G}^a(\rho)f_2(\rho,x)\}d\rho=\frac{1}{2\pi}\Psi_2\,. \tag{6.4b}$$

These equations hold for the range $L'\equiv[A,B]$. For the functions f_1 and f_2 we have the expressions

$$f_1(\rho,x)=f(|\rho-x|)+f(\rho+x)\,, \tag{6.5a}$$

$$f_2(\rho,x)=f(|\rho-x|)-f(\rho+x)\,. \tag{6.5b}$$

$\bar{G}^s(\rho)$ denotes a symmetrical charge distribution with respect to the x_2-axis and $\bar{G}^a(\rho)$ an anti-symmetrical one.

Substituting

$$\bar{G}^s(\rho)=\frac{1}{4\pi}(\Psi_1+\Psi_2)G^s(\rho) \tag{6.6a}$$

$$\bar{G}^a(\rho)=\frac{1}{4\pi}(\Psi_1-\Psi_2)G^a(\rho) \tag{6.6b}$$

in (6.4), we now obtain the equations

$$\lim_{\delta\to 0}\int_{L'-L_\delta}G^s(\rho)f_1(\rho,x)d\rho=1\,, \tag{6.7a}$$

$$\lim_{\delta\to 0}\int_{L'-L_\delta}G^a(\rho)f_2(\rho,x)d\rho=1\,. \tag{6.7b}$$

Hence two independent equations are established governing, respectively, a symmetrical and anti-symmetrical vibration. The total charges associated with the distributions $G^s(\rho)$ and $G^a(\rho)$ are denoted by Q^s and Q^a. Representing the dimensionless charge on the electrode $-B\leqq x\leqq -A$ by Q_1, and that on the remaining electrode by Q_2, we derive from (6.3) and (6.6)

$$Q_1=\frac{1}{4\pi}\{(Q^s+Q^a)\Psi_1+(Q^s-Q^a)\Psi_2\}\,, \tag{6.8a}$$

$$Q_2=\frac{1}{4\pi}\{(Q^s-Q^a)\Psi_1+(Q^s+Q^a)\Psi_2\}\,. \tag{6.8b}$$

Alternative forms of (6.8) are

$$\Psi_1=\frac{\pi}{Q^sQ^a}\{(Q^a+Q^s)Q_1+(Q^a-Q^s)Q_2\}\,, \tag{6.9a}$$

$$\Psi_2=\frac{\pi}{Q^sQ^a}\{(Q^a-Q^s)Q_1+(Q^a+Q^s)Q_2\}\,, \tag{6.9b}$$

and

$$Q_1=\frac{1}{Q^s+Q^a}\left\{\frac{Q^sQ^a}{\pi}\Psi_1+(Q^s-Q^a)Q_2\right\} \tag{6.10a}$$

$$\Psi_2=\frac{1}{Q^s+Q^a}\{(Q^a-Q^s)\Psi_1+4\pi Q_2\}\,. \tag{6.10b}$$

We now distinguish three important cases. Case I: Ψ_1 and Ψ_2 are prescribed. Charges Q_1 and Q_2 then follow from (6.8). The resonant frequencies are those values of Ω for which Q^s or Q^a becomes infinite. Case II: Q_1 and Q_2 are given. Then the potentials Ψ_1 and Ψ_2 are determined by (6.9) and resonance occurs if Q^s or Q^a vanishes. Case III: The charge of one pair of electrodes and the potential of the remaining pair are given, for instance Ψ_1 and Q_2. Applying (6.10), Q_1 and Ψ_2 are evaluated. The condition for resonance now reads that Q^s+Q^a vanishes.

To solve the integral equations (6.7) the approach described in the foregoing section is employed. Hence the interval $[A, B]$ is divided into sub-intervals separated by points ϱ_m such that

$$\rho_{2m} - \rho_{2m-1} = \rho_{2m+1} - \rho_{2m} = t_m\,, \qquad m = 1, \ldots, M\,, \tag{6.11}$$

with

$$\rho_1 = A\,, \quad \rho_{2M+1} = B\,,$$

$$t_m = \frac{A-B}{4}\left\{\cos\frac{\pi m}{M} - \cos\frac{\pi(m-1)}{M}\right\}. \tag{6.12}$$

If sets of equations of the form (5.4) are solved, the values $G^s(\rho_n)$ and $G^a(\rho_n)$, $n=1, \ldots, 2M+1$, can be computed and then the charges Q^s and Q^a according to (5.8). The coefficients of $G^s(\rho_n)$

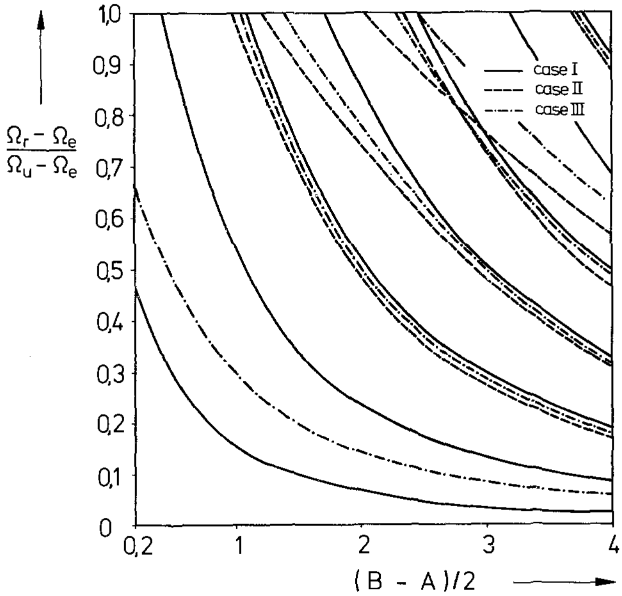

Figure 2. Resonance spectrum for $A=0.2$ and $k=0.66$.

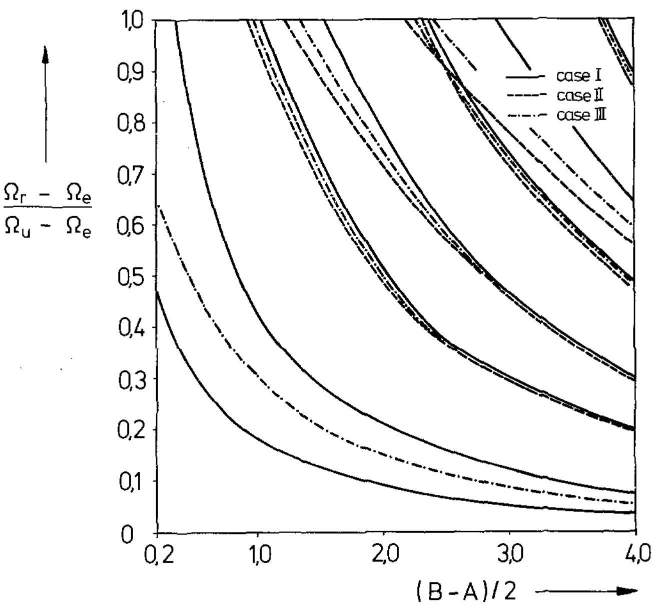

Figure 3. Resonance spectrum for $A=0.5$ and $k=0.66$.

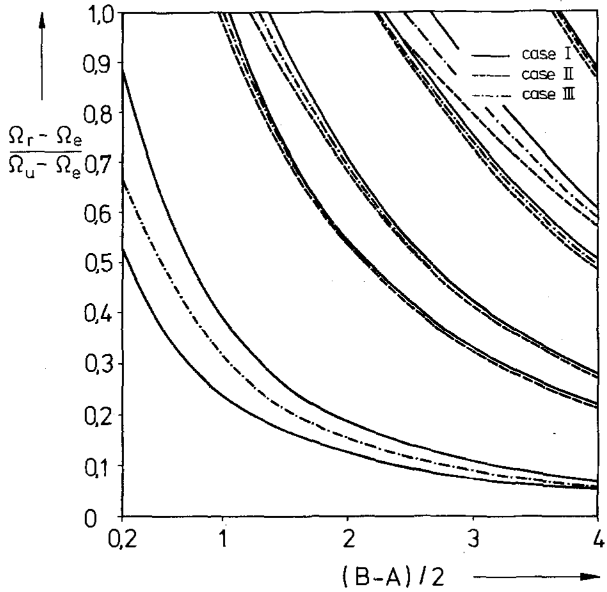

Figure 4. Resonance spectrum for $A=1$ and $k=0.66$.

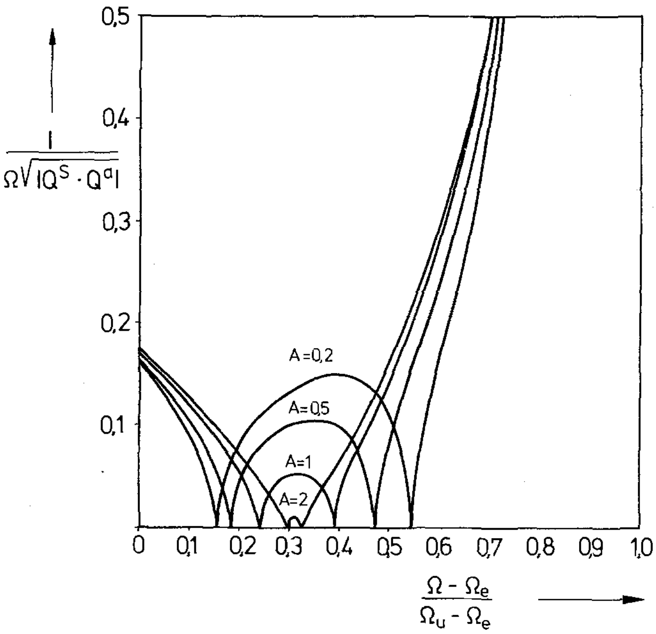

Figure 5. Characteristic impedance for $A=0.2$, 0.5, 1 and 2, $\frac{1}{2}(B-A)=1$ and $k=0.66$.

and $G^a(\rho_n)$ are infinite series of the form (5.5). These series are truncated such that Q^s and Q^a are computed with three significant digits. Computations are performed for fixed values of k, A and B. Q^a and Q^s are evaluated for a number of values of Ω in the trapped energy range. Resonant frequencies for the three mentioned cases are obtained by determining the zeros of $(Q^a)^{-1}$, $(Q^s)^{-1}$, Q^a, Q^s and Q^a+Q^s from the computed values. In the figures 2, 3 and 4 resonance spectra are given for $k=0.66$ and, respectively, $A=0.2$, 0.5 and 1. The ordinate is the quotient $(\Omega_r-\Omega_e)/(\Omega_u-\Omega_e)$; Ω_r denotes the resonant frequency, Ω_e the cut-off frequency for a fully electroded plate and Ω_u this frequency for a plate without electrodes. From (3.26) we derive $\Omega_e=0.3917$; as mentioned already, $\Omega_u=0.5$. The resonances are plotted as a function of the ratio width electrodes over thickness plate, i.e. $\frac{1}{2}(B-A)$.

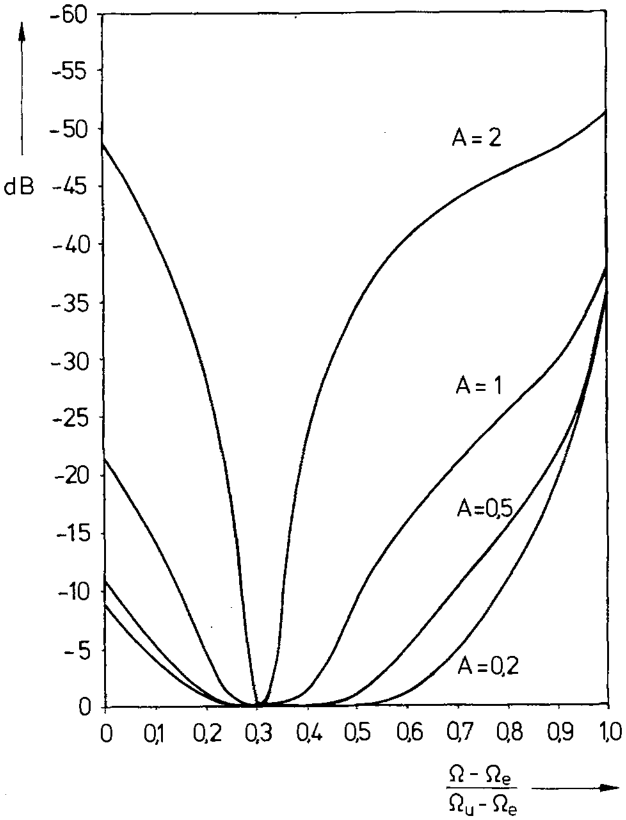

Figure 6. Insertion loss in dB for $A=0.2$, 0.5, 1 and 2, $\frac{1}{2}(B-A)=1$ and $k=0.66$.

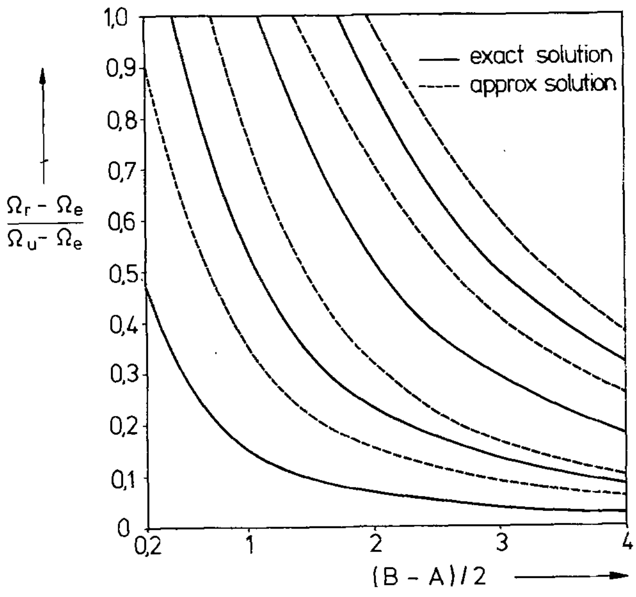

Figure 7. Exact and approximate resonances for case I with $A=0.2$ and $k=0.66$.

We observe that pairs of resonance curves can be indicated for cases I and II. One curve of a pair represents resonance of the symmetrical mode and the other one resonance of the anti-symmetrical vibration. For increasing values of A these curves tend towards each other. In order to prévent the occurrence of overtones, the width of the electrodes must be smaller

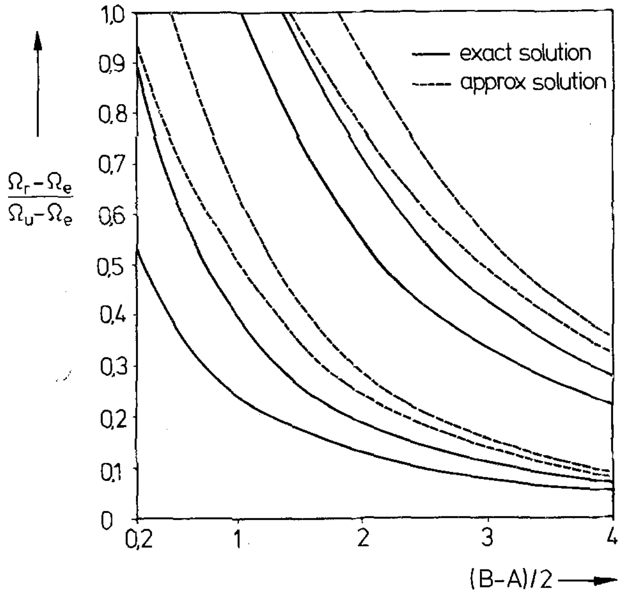

Figure 8. Exact and approximate resonances for case I with $A=1$ and $k=0.66$.

than the thickness of the plate for prescribed potentials on the electrodes. By selecting a suitable value of A, the bandwidth desired can be obtained.

Electrical engineers are also interested in the characteristic impedance and the insertion loss. The characteristic impedance is given by

$$\frac{h\mu^{\frac{1}{2}}}{\pi\, {}^{S}\varepsilon_{11}({}^{D}c_{1313})^{\frac{1}{2}}\,\Omega(Q^s Q^a)^{\frac{1}{2}}}\,. \tag{6.13}$$

In figure 5 we have plotted the dimensionless quantity $\Omega^{-1}|Q^s Q^a|^{-\frac{1}{2}}$ for several cases. The characteristic impedance is real between the zeros and purely imaginary in the remaining ranges. In figure 6 the insertion loss, which is a non-dimensional function of the frequency, is plotted, with a resistance equal to the maximum characteristic impedance between the zeros.

Figures 7 and 8 show the fundamental resonances and first overtones for case I due to the exact approach and the approximate one, described in [6]. We observe that rather large differences occur. Finally; we give in the following table resonant frequencies for case I for several values of k, with $A=\frac{4}{3}$ and $B=\frac{14}{3}$.

k	$(\Omega_r-\Omega_e)/(\Omega_u-\Omega_e)$			
0.3	0.410	0.810		
0.4	0.325	0.560		
0.5	0.255	0.385	0.980	
0.6	0.200	0.270	0.735	0.970
0.7	0.150	0.185	0.595	0.705

Remark

The approach described in sections 4 and 5 has also been applied to the problem discussed in [7], which was solved earlier in a different way. When the range [0.1] is divided into 16 sub-intervals, the results obtained now agree with the resonances given [7] within the number of significant digits.

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